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How Can the Use of Digital Games in Mathematics Education Promote Students' Mathematical Reasoning? A Qualitative Systematic Review

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Abstract

In this article, we conduct a qualitative systematic review of studies examining the use of digital games to promote students' mathematical reasoning in primary and lower secondary schools. Digital games now have a prominent role in students' leisure time, as has mathematical reasoning in curricula around the world. This study investigates how the affordances of digital game–based learning environments (DGBLEs) are used to support students' mathematical reasoning. Through a thematic analysis, we construct five distinct themes that describe how mathematical reasoning is afforded in the DGBLEs in the reviewed studies: developing (winner) strategies, exploring an immersive environment, experimenting, designing learning games and solving tasks. By analysing the themes in relation to the reasoning and proof cycle, we found that DGBLEs primarily supported exploration, conjecturing and, to a lesser extent, justification. We conclude that students' mathematical reasoning can be achieved through DGBLEs that specifically target exploration, conjecturing and justification, and by carefully structuring students' interactions with and dialogues about the games played.

Keywords Mathematics education · Mathematical reasoning · Game · Digital games · Game-based learning · Qualitative systematic review · Thematic analysis · Affordances

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Introduction

There is a long tradition of mathematics teachers employing gameplay to support students' mathematical learning (Bright, 1983; Oldfield, 1991). However, new reinterpretations of the educational role of games are motivated by the promising possibilities of digital games (Devlin, 2011), which permeate children's leisure time today. They often play for extended periods while being deeply immersed. To understand children's engagement with games, Gee (2003) studied the learning principles used by the game design industry and how to exploit these principles in educational contexts. His seminal work has contributed to an understanding that digital games can be used to introduce new literacies and new learning opportunities in education more generally.

There has been some optimism in mathematics education regarding the use of digital games (Divjak & Tomić, 2011). Following Gee's principles, Devlin argued that video games are the ideal medium for learning mathematics, and Larkin (2015) used the principles to evaluate mathematical apps for learning. Moreover, Hainey et al. (2016) reported that digital game–based learning (GBL) approaches are frequently applied in mathematics education. Based on a survey of six hundred and eighty-four K–8 teachers, Takeuchi and Vaala (2014) concluded that, among the subjects in question, 'mathematics ranked highest, with 71% of [digital game–using teachers] reporting that digital games have been either effective or highly effective in improving their students' mathematical learning' (p. 48). This indicates a strong belief among teachers that digital games can help children learn mathematics. Indeed, in a meta-analysis of computer games as learning tools, Ke (2009) found that, across subjects, games seemed to promote players' higher-order thinking (such as reasoning) to a greater degree than the learning of facts. Her finding aligns with Gee's view on the learning potential of digital games.

Contrary to these positive findings, Young et al.'s (2012) review stressed the difficulties involved in exploiting the potential of GBL in mathematics education, as opposed to, for instance, using GBL to learn language and history. Moreover, two recent meta-analyses of GBL within the context of mathematics education only found minor positive effects (Byun & Joung, 2018) or marginally significant effects (Tokac et al., 2019) on students' mathematical achievement. These meta-analyses also reported considerable differences in the effectiveness of GBL approaches in the selected studies. Thus, the key question is not whether digital games as such are effective in mathematics teaching but, rather, how the use of specific digital games can lead to students' mathematical learning.

In this review, we seek to understand how digital games can be used to promote students' mathematical reasoning in primary and lower secondary education. Our motivation for focusing on digital games is that, while they are perceived to offer new and transformational potential in mathematics education, this has seldom been realised (Lowrie & Jorgensen, 2015). Therefore, we want to understand the specific features of the digital components in terms of their potential to support students' mathematical learning.

We identified three motivations for combining our focus on digital games with mathematical reasoning. The first is that research into the use of analogue games suggests that games can be used to help students construct mathematical arguments (Moyer & Bolyard, 2003; Olson, 2007), explore gameplay dynamics that possibly lead to reasoning and justification (Chick, 2010), generalise and reason for a winning strategy (Day, 2014) and reason axiomatically during gameplay (Kaufmann et al., 2009). The second motivation is that previous reviews on games and mathematics learning have focused broadly on mathematical conceptual knowledge or procedural skills and not on specific processes or competences such as reasoning. The third motivation is that over recent decades, mathematical reasoning has gained a prominent role in primary and secondary curricula around the world, and a number of studies have shown that mathematical reasoning is difficult for students to learn and for teachers to teach (e.g. Stylianides et al., 2017). Therefore, our primary research interest here is to understand how digital games (and the context in which they are used) can support primary and lower secondary students' learning of mathematical reasoning.

Games and Affordances

In our definition of a digital game, we mean any game (including game apps) that can be played on a digital device such as a computer, tablet, mobile device or gaming console. The sheer diversity of games makes it difficult to define a game more precisely. However, we rarely have trouble recognising a game when we see one (Skaug et al., 2020). For the purpose of our study, we understand games in line with an often-used definition: 'a game is a system in which players engage in an artificial conflict, defined by rules, that results in a quantifiable outcome' (Salen & Zimmerman, 2004, p. 80). This means that a game involves one or more active participants (players) who interact with a system and engage in an artificial conflict, that there is some form of contest and that the game maintains some boundary from everyday life. The game is structured by rules and playing it will result in a quantifiable outcome: for example, one player wins and another loses (Salen & Zimmerman, 2004), or the desirable completion of a challenge, such as the rescuing of a wounded eagle (Gresalfi & Barnes, 2016).

We understand participation in games along the lines of Goffman (1961), who distinguishes between the *game* as a set of rules and materials and the *gaming encounter* as the social interaction in which the game takes place. In this understanding, a game will not simply be played as designers intend, but will be constructed as a local cosmos with specific meaning by the participants in the gaming encounter. According to Goffman, 'while it is as players that we can win a play, it is only as participants that we can get fun out of this winning' (p. 34). The sense that is created from playing a game is thus created by the participants in this social encounter, and the goals of such encounters are not solely directed at winning. DeKoven (2013) describes that, in a play community with the aim of collective enjoyment, the goal could be to participate in the community by engaging in the game; in some role-playing games, playing to win can be considered to be an outright destructive approach (Gribble, 1994).

Deterding (2013) identifies five different motivational relevancies for playing digital games: relaxation, relatedness, engrossment, competence and achievement. Such relevancies can be seen in games such as *Minecraft* played in creative mode for engrossment, or *The Sims* played for relaxation. The need to understand students' participation in games is highlighted by Bishop (1991), who suggests that fun should be an important construct when appreciating the playing of games as a mathematically significant activity and when organising a mathematics curriculum that incorporates games. However, how to organise a curriculum in this way is an open question and the concept of fun is seldom defined in the GBL literature. Nonetheless, in our understanding of games, we stress a two-fold perspective: one related to the formal structures of the game (i.e. its design) and the other related to the social interactions of the players involved in the game.

The concept of affordance has been used in technology-related fields, such as information studies, to capture aspects of the relationship between users and technology in relation to possibilities for action. In this sense, what is afforded is considered more than part of the technology; it also emerges as possible interactions from the relationship between the user and the technology, where multiple affordances can be identified (Volkoff & Strong, 2017). Gibson (1977), who initially coined the term *affordance* in this context, defined it as an interactional relationship among objects, actors and the actions that the environment offers to specific actors. Importantly, this does not imply that these actions will necessarily occur. For GBL approaches, this means that, even though specific affordances are designed into a game, it is not certain that students consider them as possible interactions. Therefore, some students play games differently from as intended by the teacher or designers (e.g. Al-Washmi et al., 2014).

Drawing on Gibson's work, Brown et al. (2004) described affordance as a promising notion in analyses of classroom activity, '[where] students are immersed in a mathematically and technologically rich learning environment' (p. 126). Along the lines of Kress (2005), we use affordance as a concept to describe students' possibilities for action in relation to digital games and the surrounding environment with its culturally and socially produced resources. In this understanding, the affordances in a learning environment include the digital game and its specific design features, the relationships between the student and the game and the social setting, including teacher support (Tanner & Jones, 2000).

Using the phrase *afforded interactions leading to mathematical reasoning*, we refer to the relationships among students, games and the learning environment that can lead to students' engagement in mathematical reasoning. This can be interactions afforded by specific design features of the digital game, but also by the teacher's ways of organising classroom talk, group-work or tasks around the game. To capture this, we use the abbreviation DGBLE for digital game–based learning environment. Thus, we attempt to identify affordances both of the design features of digital games and of the learning environments based on a GBL approach with digital games and how a teacher's scaffolding can support students' mathematical reasoning. With this understanding of affordance, we state our research question as follows:

> How do DGBLEs afford primary and lower secondary students' learning of mathematical reasoning?

Digital Games and Mathematics Education

Despite the promising opportunities for digital games to support mathematical learning (Devlin, 2011; Ke, 2009), studies have pointed to at least three complexities. The first is whether students become more motivated to learn mathematics when they use digital games. Studies disagree on this significant question (Monaghan, 2016). A second complexity regards the use of *non-school digital games*: that is, games that are not specifically designed to be played in a mathematics lesson (e.g. *Angry Birds*, *Plants versus Zombies* or *The Sims*). Avraamidou et al. (2015) have shown that the mathematical elements in these games are mostly invisible to the player, as they are integrated into the algorithmic design of the game. The mathematics in these games also differs from the mathematics curriculum. Thus, it can be a challenge both for teachers and for students to discover the mathematics of a digital game and for teachers to establish meaningful connections between the game and the mathematics curriculum. A third complexity is the unpredictability of students' gameplay, which makes it difficult for the teacher to achieve common learning goals.

The various digital games designed to support mathematics learning introduce at least three additional challenges. The first is that the overwhelming majority of digital games used for mathematics learning only enhance basic skills (Larkin, 2015). These simple skill-and-drill games are often simple training software in attractive graphics that mainly focus on procedural fluency in subject areas such as algebra, geometry and measurement. As such, they cover only small and not very complex parts of the mathematics curriculum (Byun & Joung, 2018), such as the four basic operations (de Carvalho et al. 2016). Some researchers perceive such games as a waste of students' time (Fox et al., 2000), criticising them for not engaging students in mathematical processes, such as problem solving and reasoning (van Eck, 2015).

A second challenge is that the available empirical research is rather narrow in terms of mathematical content areas and builds predominantly on these skill-and-drill games or gamified versions of traditional worksheets. A third challenge is a concern raised by Lowrie and Jorgensen (2015), that these visually appealing drill-and-practice games seem to be the kind of games that teachers will use. This concern is underscored by two international teacher surveys: Egenfeldt-Nielsen et al. (2011) show that 52% of a sample of two hundred and seventy-five teachers used games designed to learn skills, while Takeuchi and Vaala (2014) stress that mathematics teachers generally used short stand-alone games or mini-games. Evidently, these teachers did not use longer, complex, immersive, decision-making games, even though these in general perform better in research on learning outcomes. However, in contradiction to these challenges and tendencies, other studies (e.g. Ke, 2009) point out that, in general, games seem more useful in promoting higher-order thinking, such as reasoning, than they are in supporting conceptual and factual knowledge acquisition.

Reasoning in School Mathematics

In curricular documents around the world, reasoning has gained a prominent role in school mathematics during the last two decades. To use the Danish syllabus as an example, the so-called mathematical thinking and reasoning competency (Niss & Højgaard, 2019) is expected to play an important role from the first to the ninth grades in all sub-topics of mathematics (Ministeriet for Børn, Undervisning og Ligestilling, 2015). This competency encompasses being 'able to relate to and pose the kinds of generic questions that are characteristic of mathematics and relate to the nature of [expected] answers [… as well as] to analyse or produce arguments (i.e. chains of statements linked by inferences) put forward in oral or written form to justify mathematical claims' (Niss & Højgaard, 2019, pp. 15, 16). A similar shift in focus can be seen in the Common Core State Standards in the USA (CCSSM, 2010), where two of the eight mathematical practices involve reasoning: that is, the ability to reason abstractly and quantitatively, to construct viable arguments and to critique the reasoning of others (e.g. Dingman et al., 2013).

This renewed focus on mathematical reasoning has at least three possible explanations. The first is that reasoning and proving are key activities in the field of mathematics. As Schoenfeld (2009) states, 'If problem solving is the "heart of mathematics", then proof is its soul' (p. xii). The expectation is that students can learn to think mathematically and learn mathematics in a more in-depth manner through engagement in activities that involve reasoning and proving (Stylianides, 2016). The second includes the new opportunities for explorations, dynamic visualisations, calculations and so on, offered by digital technologies (e.g. dynamic geometry systems and computer algebra systems).

These technologies allow students to explore and discover relationships, systems and regularities, and thus to formulate hypotheses more easily (Sinclair & Robutti, 2013). The third explanation is a general tendency in mathematics education also to focus on students' learning of mathematical processes, such as reasoning, modelling and problem solving, rather than (as previously) on their learning of mathematical products, such as concepts and procedures (NCTM, 2000). The idea is that students will learn these products more profoundly by engaging in mathematical processes.

Based on the above-described tendencies and explanations, this study will focus on students' learning of one specific process, namely mathematical reasoning, and not on their learning of mathematical products. However, there is overwhelming evidence that teachers find it challenging to teach mathematical reasoning and that students find it difficult to learn (Nardi & Knuth, 2017; Stylianides, 2016; Stylianides & Stylianides, 2017). In the case of Denmark, these difficulties were recently documented in the large-scale project on student mathematical reasoning. It was called *Quality in Teaching Danish (L1) and Mathematics* (Hansen et al., 2020), where teachers struggled to teach mathematical reasoning even though they used tasks predesigned by researchers for this purpose. The learning results were poor in terms of student mathematical reasoning (Larsen & Lindhardt, 2019).

In the field of mathematics education, mathematical reasoning and proofs are defined in various ways (Stylianides, 2016). Stylianides et al. (2017) identified a range of definitions from a mathematical stand-point that associate reasoning and proofs with logical deduction, and link premises with conclusions to a social perspective that focuses on how members of a community jointly approve an argument as a proof. For our purpose, we found the definition of reasoning and proof—in terms of *a reasoning and proof (R&P) cycle*—proposed by NCTM (2008) appropriate. In contrast to the mathematical thinking and reasoning competency in the Danish national syllabus, the R&P cycle emphasises the meaning-making aspects of the reasoning process, including a broader set of activities. This broader approach to reasoning seems to be more in line with the opportunities for actions that may arise when students play digital games.

The R&P cycle consists of three interconnected phases: exploration, conjecturing and justification. In the exploration phase, students intuitively investigate mathematical phenomena for patterns and structures, in an attempt to detect regularities and relationships. In the conjecturing phase, they use their observations to formulate hypotheses or modify existing hypotheses. In the justification phase, they validate or refute their conjectures by explaining why they are true (or untrue) from a mathematical point of view. The result of the justification phase can be a mathematical proof in a formal sense, but it can also consist of more informal arguments, dependent, among other things, on the age of the students. The three phases can occur in any order, and reasoning processes in classrooms often consist of iterative cycles of these phases. Here, we apply this approach to reasoning in school mathematics to investigate the affordances of DGBLEs.

Methodological Approach

We conducted a qualitative systematic review and used a systematic literature search strategy to synthesise the existing research through a thematic analysis (Grant & Booth, 2009). To review a broad spectrum of research and follow a systematic approach, we selected and searched four relevant academic databases: Education Research Complete, British Education Index, ERIC through Ebscohost and Matheduc (a mathematics education database). We followed the building blocks search strategy (Johannsen & Pors, 2013) by identifying key terms and including synonyms for each of them separated by the Boolean operator OR. The four key terms, including synonyms, were separated by the Boolean operator AND. Our first search string consisted of the four key terms (including synonyms): *game*, *reasoning*, *mathematics* and *age group*. We selected studies where all four terms (or synonyms, see Table 1) were present in the title, keywords or abstract.

Following Sidenvall (2019), we pilot-tested the search string and found that the inclusion of more than the key term—*reasoning*—in the second search term resulted in too many unwanted results. Therefore, we removed the synonyms (*problem solving*, *modelling* and *higher-order mathematical thinking*) in the second term and only used *reasoning*, which encompassed prefixes such as *mathematical*, deductive and *inductive*. We did not include more specific features of reasoning, such as exploring, conjecturing, justifying, generalising or identifying patterns (Jeannotte & Kieran, 2017). Another limitation for the search was to include only peer-reviewed studies written in English.

After the removal of forty-one duplicates found in several databases, our final search string produced a hundred and fifty-one singular results. We based our first screening of the studies' titles and abstracts on four criteria to ensure a systematic and consistent inclusion process. The inclusion criteria were as follows:

Table 1 Final search string

Term 1	Term 2	Term 3	Term 4
game OR gaming OR gamification OR game-based learning OR educational games OR puzzle	reasoning	mathematics OR math* OR mathematical	‘basic education’ OR K12 OR primary OR secondary OR elementary OR ‘middle school’

1. An explicit focus on student reasoning
2. Empirical research studies
3. A broad focus on students, rather than specific groups of students
4. Publication in peer-reviewed journals, books or conference proceedings

The first screening resulted in twenty-five studies. For some studies, we could not determine whether they fulfilled the criteria simply from reading the title and abstract. If not, they were excluded from the subsequent steps. We also read through the references in these studies for titles relating to our research question and screened possible relevant studies according to the four criteria. This resulted in six additional studies. Thus, we selected thirty-one studies in total for full-text reading. Based on this reading, we excluded seventeen studies for various reasons: for example, some did not address digital games, some focused on the tertiary level of education and others were not empirical studies. In the end, fourteen studies matched our criteria and were selected for review. This small number of studies should be seen in relation to the fact that relatively few studies exist on games and learning in mathematics (Ke, 2014). Aside from the strategy games in Houssart and Sams (2008) and in Lee and Chen (2009), the digital games in the remaining twelve studies were designed specifically for the purpose of supporting students' mathematical learning.

Thematic Analysis

In the fourteen selected studies, we found mathematical reasoning to be connected to a wide range of interactions, including engaging with a narrative (Gresalfi & Barnes, 2016), designing a game (Ke, 2014) and discussing a single move in a game of *Lines* (Houssart & Sams, 2008). This diversity in DGBLEs across the studies—both in terms of the game designs and of the didactical framing of activities and dialogue about the gameplay—prompted us to use a thematic analysis (Braun & Clarke, 2006). We used their six steps to structure our thematic analysis. Firstly (step 1), we familiarised ourselves with the data through full-text readings. Secondly (step 2), we generated the following initial codes: student age, methodological approach, description of digital game, kind of student reasoning, emphasised aspects of (design of) teaching and perceived affordances of the digital game.

We then analysed each study using these codes. Later in the process, we also coded the studies according to author affiliation and research interest, as well as the research fields of the journals. Thirdly (step 3), we developed themes to capture a patterned response or meaning in the DGBLEs in order to understand the opportunities emerging in the studies for learning to reason mathematically. The affordances in the DGBLEs led us to divide the studies into five themes, each comprising distinct potentials in and challenges to establishing a learning environment where mathematical reasoning was afforded. These themes were continuously reviewed (step 4), then refined and renamed (step 5) in order to ensure clear definitions of each theme. Finally (step 6), examples were found to represent each theme.

The themes describe the overall forms of interaction in the different DGBLEs in terms of the kinds of mathematical reasoning afforded to the students. We call this *the main affordance for mathematical reasoning of a DGBLE*. For example, we coded one affordance for mathematical reasoning under the theme 'exploring an immersive environment'. Different elements of the DGBLE afforded exploration of the immersive environment, one of which was a *non-player character* (NPC) who gives feedback to students' recommendations, requiring them to apply statistical reasoning to determine their consequences in the gameplay (Gresalfi, 2015).

Another type of main affordance for mathematical reasoning was coded under the theme 'developing (winner) strategies', an example of which was provided by Houssart and Sams (2008). Here, the computer is the students' opponent in a strategic game, and the students position the computer as an adversary whom they have to outsmart and after whom they have to model their own winning strategies in terms of how the computer-player played. The specific ways in which the computer-player formed part of this DGBLE thus afforded the students to reason through the development of winner strategies.

The five themes considered are as follows.

1. *Developing (winner) strategies*—the main affordance for mathematical reasoning is to figure out how to play better than one's opponent(s) by developing (winner) strategies.
2. *Exploring an immersive environment*—the player is invited to explore a virtual world and engage in solving problems within it. The main affordance for mathematical reasoning is to solve problems and understand the consequences for the world, scenario or narrative.
3. *Experimenting*—the player is offered an experimental context where the main affordance for mathematical reasoning is to experiment with different values of game settings and explore their relationship.
4. *Designing learning games*—the main affordance for mathematical reasoning is to design and create content-specific learning games.
5. *Solving tasks*—the main affordance for mathematical reasoning is to solve closed tasks to move forward in an appealing game context.

Finally, we analysed the studies according to whether (and, if so, how) they addressed each of the three R&P phases. This analysis showed a variety of ways in which the use of digital games afforded students' engagement in different parts of the mathematical reasoning process.

The following five sub-sections present the main affordances for mathematical reasoning. Each contributes to answering our research question by describing and exemplifying different ways in which DGBLEs afforded students' mathematical reasoning through the lens of the R&P cycle.

Developing (Winner) Strategies

The main affordance for mathematical reasoning by the four studies under this theme (Houssart & Sams, 2008; Lee & Chen, 2009; Pareto, 2014; Pareto et al., 2012) was to develop one or more (winner) strategies to outplay one's opponent(s). This affordance was provided by using strategic games in which the objects of mathematical reasoning were closely tied to their rules and aims, and students were required to explore and identify underlying patterns and develop strategic approaches to win the games. We emphasise two overall findings here.

The first finding was that the studies did not consider students' interactions with the games as sufficient in themselves to support their learning. Rather, the game was used as part of a wider DGBLE, where students' gameplay and situations around their gameplay were scaffolded. Learning was thus afforded both by the design features of the game and by the environment in which it was played. The didactical framing in the environment included teachers posing specific questions that related game aspects to mathematics (Lee & Chen, 2009), as well as the use of an intelligent *teachable agent* (TA) that challenged students' arguments (Pareto, 2014; Pareto et al., 2012). The teacher's scaffolding of group and classroom talks in Houssart and Sams (2008) provides an example of framing dialogues about gameplay that led students to conjecture and then modify and justify their conjectures.

Houssart and Sams (2008) was the only reviewed study that convincingly engaged students in all three R&P phases. It involved nine primary classes in England (children aged 9–11 years) in the use of a game of strategy called *Lines* (in four to eight lessons). The game is a digital version of *Connect Four*, but its pieces can be placed anywhere on the digital board and the co-ordinates of each move are displayed along with sequences of previous moves. In the study, groups of two or three students played against the computer. The computer adopted the best strategy, but since the students were always allowed the first move, it was possible for them to develop a winning strategy and win.

A key factor in the study was the 'set of rules for talk' (p.60)—inspired by Mercer's (1995) *exploratory talk*—that each class developed to encourage students to work and talk together while using computers. The authors reported that questions such as 'Where should we put our counter?', followed by 'Why do you think that?', enabled the students to explore and reason together in ways that led them to develop winner strategies, which they might not have been able to achieve without being taught how to reason effectively together (p.69). Another key factor in the study was the possibility of rewinding the game (i.e. a design feature), so that the combined set of moves could be examined by the students.

A second finding was that three of the four studies used a virtual character—a specific affordance of digital games—to support student mathematical reasoning. Such a character can be an expert game player that inspires students to develop winner strategies by modelling its behaviour or encouraging students to outsmart the computer (Houssart & Sams, 2008). A promising example of a designed feature added to a game is the virtual character in the form of a TA (Pareto, 2014; Pareto et al., 2012), an agent that can be taught to play the games by the students through a master–apprentice scheme.

These two studies used a collection of two-player digital games built on an animated, graphical model to represent numbers and simulate arithmetic calculations. The game objectives were to teach younger students arithmetic concepts (especially place value and operations involving carrying and borrowing) and to stimulate their reasoning and strategic thinking. The students and computer-player took turns placing cards representing an arithmetic operation on a gameboard until all cards were played. In one game, the students placed cards on top of already-played cards and received points for each ten they created. The students could see their opponent's cards: thus, they could prevent their opponent from creating tens and develop strategies to avoid this.

To win, students had to develop strategies by calculating mentally, by reasoning about numbers and computations and by judging the impact of certain moves on future game conditions. However, these were not explicit activities; rather, the game was purposely designed so they would appear as side effects of playing (Pareto et al., 2012). The TA learned about the game in two ways. One was to play against students and learn from their corrective feedback and acceptance/rejection of the TA's suggestions. The other was to watch students play and ask them about the reasons for their choice of cards before the consequences of these choices appeared in the game. For example, the TA could ask 'Why do you choose card seven?' in a game where the goal is to make tens, with the following possible answers:

- 'It's obviously the best one! It gives 1 point and it's the only card that blocks the opponent'.
- 'It gives 1 point. Unfortunately, it doesn't block the opponent from getting points'.
- 'It doesn't give any points, but it blocks at least the opponent from getting points'.
- 'I don't know'. (Pareto, 2014, p. 262)

To choose from such pre-defined explanations, students must evaluate them in relation to the game situation. In this sense, they can be said to be engaged in the justification phase, albeit without being required to develop arguments themselves. One result is that students who improved their performance based on the TA questions became better at selecting the correct explanations, but not necessarily at developing such explanations themselves (Pareto, 2014). The purpose of the TA was both to challenge students to reflect on their gameplay and to act as a role model of exploration, by prompting students to consider and explain their choices. As the students taught the TA to play, the TA's learning level became a proxy for their own learning.

Overall, this theme shows that students' engagement in one or all three R&P phases could be afforded in a DGBLE with a strategic game, if the students' interactions with the game were framed didactically. This could be achieved either through specific rules to support explorative talk or, to some extent, by introducing virtual characters that either played the game or simulated learning the game from the students.

Exploring an Immersive Environment

The main affordance for mathematical reasoning by the four studies under this theme (Gresalfi, 2015; Gresalfi & Barnes, 2016; Ke, 2019; Wijers et al., 2010) was to explore an artificial virtual world and solve problems within it: for instance, to help an injured eagle in *Adventure at Boone's Meadow* by planning flight routes (Gresalfi & Barnes, 2016) or to help rebuild an area hit by a natural disaster in an architecture simulation game (Ke, 2019). The affordances for mathematical reasoning in these DGBLEs were closely tied to the extent to which the immersive environments carried out the contextual consequences of the player's choices. In cases of high and credible alignment between the player's choice and its contextual consequences, the players (i.e. students) had better opportunities to evaluate, discuss and argue about the appropriateness of their solutions from both a game and a mathematical perspective. We stress three findings here.

The first finding is that exploring an immersive environment can afford the need to justify solutions mathematically, as long as justifiable solutions are required by the environment. One example of this was when students assisted a mayor in the virtual *Ander City* to use statistical problem solving to make decisions about which bike brands were safer to use (Gresalfi, 2015). By comparing sets of data using statistical tools (e.g. mean, median), the students developed data-based arguments for or against specific decisions. The consequences of their recommendations were then enacted in the game environment. An NPC provided the students with statistical data, as well as feedback on their recommendations, which afforded them to relate the statistical reasoning behind their recommendations to the context where the consequences appeared.

During a first research iteration, Gresalfi found that asking students to explain their recommendations was rarely sufficient to provoke discussions about statistical tools or prompt mathematical argumentation, as students seemed to think that the problems only had one legitimate solution. Therefore, a second research iteration was conducted, where a competing mayor was implemented to challenge the students' recommendations and arguments, and to offer conflicting perspectives. This helped the students understand that alternative solutions were possible, which improved the justifications for their recommendations.

The second finding is that immersive environments can afford multi-faceted, contextual perspectives on problems through the enactment of contextual consequences of their solutions. This is emphasised as a key ingredient in promoting students' justification, as it supports them in realising the possibility of different solutions, which, in turn, affords them to justify their own solution. One example is *Adventure at Boone's Meadow* (Gresalfi & Barnes, 2016), in which students must rescue a wounded eagle by making transportation-related decisions. Depending on their solution, the eagle will live as before, have to endure with an amputated wing or die.

As the consequences of the decision appear in the game (which the authors called consequential feedback), the students can evaluate and validate their mathematical thinking in relation to this context. If, for instance, the eagle dies, their solution may involve incorrect calculations or ignore important contextual features that should have been included in their calculations or reasoning. Here, the authors distinguish between mathematical justifications (e.g. use of mathematics to justify a decision, for example, by comparing route lengths) and consequential justifications (e.g. justifying one's decision in relation to its contextual consequences). This signals that not all the students' justifications in this environment were mathematical (and hence why we only marked this study with (X) under justification).

The third finding is that the timing of consequential feedback provided through and in relation to the game narrative was essential to support the students' mathematical reasoning, and was especially effective in the explorative and conjecturing phases. They found that feedback provided after students' problem solving was not used in revising their mathematical reasoning or in considering alternative solutions. However, consequential feedback provided early in the students' thinking (about how to solve a problem) nurtured their awareness of multiple perspectives on the problem, potentially factored into their thinking about aspects they would not have otherwise considered and promoted a higher degree of justification for their final solution. Additionally, early consequential feedback on the students' initial hypotheses enabled them to test these hypotheses and realise that other solutions existed.

Experimenting

The main affordance for mathematical reasoning in the three studies under this theme (Bakker et al., 2015; Kolovou & van den Heuvel-Panhuizen, 2010; van den Heuvel-Panhuizen et al., 2013) was to experiment with different values of game settings and explore their relationship. One example was the game *Hit the Target*, designed by the research team behind the three studies, which is an interactive simulation of an archery game consisting of a bow, a target, a pile of arrows, rules for distribution of points and a scoreboard. Students can change the number of arrows and points given for a hit and a miss and, by experimenting with these quantities, they can realise the co-variation between them and the score number.

In the three studies, the research team investigated the influence of different conditions on students' algebraic reasoning, such as playing at home or at school with different didactical framings. Three key points are highlighted here. The first is that playing an experimental game at home can be an effective extension of school learning. Students are reported to perform better in the on-line environment, via game-generated and situation-based feedback, than in a paper-and-pencil scenario without feedback (Kolovou & van den Heuvel-Panhuizen, 2010). However, statistical analysis of results from a randomised, longitudinal experiment showed no significant effects for students playing at home without the support of didactical framings around the game (Bakker et al., 2015).

This implies that simply playing at home is not necessarily sufficient to facilitate student mathematical reasoning. Rather, students' gameplay needs to be directed at specific problems or accompanied by activities that can elicit the discovery of embedded mathematical relations in the game (van den Heuvel-Panhuizen et al., 2013). These activities can be in the form of follow-up class discussions at school on student discoveries of certain relationships, systems or patterns when they play (Bakker et al., 2015) or non-routine tasks that require students to experiment with game settings to solve them, such as 'What is the game rule to get 16 points in total with 16 hits and 16 misses?' or 'Are there other game rules to get 16 hits, 16 misses, and 16 points?' (van den Heuvel-Panhuizen et al., 2013, p. 290).

The second point is that digital games inviting students to experiment with quantitative values of game settings, and to collect data on the effect of these changes, have the potential to engage students in the exploration and conjecturing phases. Furthermore, the three studies indicated a progression whereby younger students primarily engaged in the exploration and conjecturing phases, while students in the fifth and sixth grades also engaged in the justification phase. Justification consisted primarily in arguing for experimental results by performing calculations.

The third point is that an experimental game should offer students situation-based feedback on their actions, rather than evaluating their answers, as such feedback enables students to revise and refine their solutions and reflect on their learning process (Kolovou & van den Heuvel-Panhuizen, 2010). Situation-based feedback was provided indirectly by hitting the target in the score number, which was influenced by the students' choices of the number of arrows and points given for a hit and a miss. In this sense, the feedback invited the students to explore the influence of varying the game settings to solve the tasks involved.

Designing Learning Games

The main affordance for mathematical reasoning in the one study under this theme (Ke, 2014) was to design and create content-specific learning games. This was envisioned as engaging students in mathematical reasoning in three ways. The first was by providing opportunities to learn mathematics by designing a game with the aim of explaining content-specific concepts or processes to younger students. The second was by working with mathematical concepts, such as variables, when programming a game. The third was by considering how mathematical concepts affected the design of the game (Ke, 2014). This theme is distinct from the others, because its main affordances for mathematical reasoning are related to creating games and not playing them. The kind of reasoning addressed in the game creation addressed exploration, conjecturing and justification, but this was in relation to the interplay between developing a gameworld and using mathematics (e.g. numbers, x- and y-coordinates, variables) to program it.

The first finding is that it was challenging for students to design a game about mathematics, even when they had been exposed to a variety of such learning games. Ke (2014) described how the mathematics in the students' games were predominantly depicted as integer calculations and as an extrinsic add-on to game actions. Some games included no mathematical learning content at all. The second finding is that it was challenging for students to base their programming on abstract and quantitative reasoning. Ke found that, rather than analysing scripts or coding gameplay actions, the students engaged in aesthetics programming, such as designing game sprites.

Furthermore, they explored their use of variables in programs by re-executing their games without first exploring the relationship between such variables or engaging in deeper learning about them as a mathematical concept. Therefore, rather than exploring the mathematical content in the games and the gameplay mechanics, the students spent time and effort creating gameworlds and crafting their story using visual and sound effects. Only a few students reasoned abstractly as they tried to model and co-ordinate the animations and events in a game.

Solving Tasks

The main affordance for mathematical reasoning in the two studies under this theme (Wouters et al., 2015, 2017) was to solve traditional closed tasks in order to move forward in an appealing game context. The studies examined the effect of curiosity-triggering events (Wouters et al., 2015) and surprising events (Wouters et al., 2017) in the game *Zeldenrust* in secondary pre-vocational education. The game is a simulation of a summer job at a virtual hotel. It integrates a set of proportional reasoning problems related to different authentic situations, such as sorting a refrigerator and mixing drinks.

The studies investigated *proportional reasoning skills*, defined as the activities students perform when they find solutions to comparison problems, missing-value problems and transformation problems (Tourniaire & Pulos, 1985). Wouters et al. (2015) presented an example task for measuring proportional reasoning skills: 'For a banana milkshake, you have to use 28 bananas and 48 units of ice. How many units of ice do you need if you are going to use 56 bananas and you want to remain the same proportion?' (p. 197).

Both studies (Wouters et al., 2015, 2017) showed that students improved their proportional reasoning skills. However, no evidence was found of students transferring their reasoning to problems in different contexts, becoming more motivated (in fact, the opposite was observed)or developing new mathematical insights (Wouters et al., 2017). The authors attributed these results to the strongly repetitive nature of the game (Wouters et al., 2015), indicating that continuously using repetitive games, which resemble training software in the guise of a game, was not valued by the students—at least not at this age.

Discussion

The mathematical content areas in the studies were varied and included the following: arithmetic (Pareto, 2014; Pareto et al., 2012), early algebra (Kolovou & van den Heuvel-Panhuizen, 2010; van den Heuvel-Panhuizen et al., 2013), statistics(Gresalfi, 2015), fractions (Gresalfi & Barnes, 2016), geometry (Ke, 2019; Wijers et al., 2010) and co-ordinates and variables (Ke, 2014). Across these different content areas, the thematic analysis identified five distinct ways that DGBLEs afforded students' mathematical reasoning. We now discuss these results. First, we synthesise the results by answering our research question. Second, we highlight promising game design features for mathematical reasoning. Third, we discuss how the studies differed in their ways of defining mathematical reasoning. Fourth, we discuss issues with the documentation of student mathematical reasoning. Fifth, we note obstacles to supporting students' mathematical reasoning in the studies.

Affordances for Mathematical Reasoning

The thematic analysis answered our research question by identifying five general ways in which students' mathematical reasoning was afforded in the selected studies. Now, we synthesise and discuss how, and to what extent, the DGBLEs afforded students' mathematical reasoning in terms of the R&P cycle, which consists of the following three phases: exploration, conjecturing and justification.

As Table 2 shows, eleven (twelve including the markings in parentheses) studies engaged students in exploration, five (ten) studies engaged students in conjecturing and one (seven) study engaged students in justification. This means that the DGBLEs afforded new learning opportunities that particularly supported students' exploration and conjecturing and, to a lesser extent, their justification. This seems to reflect a level of complexity in terms of what is required to engage students in mathematical reasoning: exploration is the least complex, while justification is the most complex. In addition, none of the studies engaged students in formal proving, which is arguably the most complex form of justification. This indicates that DGBLEs are particularly suited to afford exploration, conjecturing to a certain degree and justification to a lesser degree.

In terms of exploration, all studies except those under the *solving task* theme afforded the opportunity, to some extent, for students to explore mathematical relationships or relationships between mathematics and the game contexts. Under the theme of *developing (winner) strategies*, the students explored and identified the underlying patterns of the rules in the games. Under the theme of *exploring an immersive environment*, they explored how the environment carried out the consequences of their choices, while under the theme of *experimenting*, the students explored the relationships between different values in game settings.

Concerning conjecturing, the five studies (marked with an X) under the first three themes (*developing (winner) strategies, exploring an immersive environment* and *experimenting*) afforded students the opportunity to make conjectures of a mathematical character. According to these studies, it is crucial to scaffold students' conjecturing in relation to their gameplay; they provided no evidence that students' conjecturing occurred as a result of simply playing these games. Houssart and Sams (2008) and Bakker et al. (2015) have stressed the importance of the teacher's scaffolding of explorative talks among students and debriefing sessions for the whole class to encourage students to formulate their own hypotheses and talk about their discoveries of new strategies or relationships among variables or quantities. As an alternative, Gresalfi (2015) and Gresalfi and Barnes (2016) exploited a digital design feature in the form of a virtual character that, among other things, supported the students in formulating hypotheses. However, it is less clear whether the students' conjectures were related to statistics or to the anticipated consequences of their choices in virtual environments. In Pareto (2014) and Pareto et al. (2012), a TA afforded the students with an opportunity to engage in conjecturing, primarily by assessing the TA's own conjectures and questions.

In relation to justification, Houssart and Sams' (2008) study provided the most convincing case of students' engagement in mathematical argumentation and, thus, in the entire R&P cycle. The students in this study argued mathematically as to why

Table 2 Overview of thematic analysis

	Main Affordance forMathematical Reasoning					Forms of Mathematical Reasoning, R&P Cycle		
	1. Developing (winner) strategies	2. Exploring an immersive environment	3. Experimenting	4. Designing learning games	5. Solving tasks	Exploration	Conjecturing	Justifying
Pareto (2014)	X					X	(X)	(X)
Pareto et al. (2012)	X					X	(X)	(X)
Houssart & Sams (2008)	X					X	X	X
Gresalfi (2015)		X				X	(X)	(X)
Gresalfi and Barnes (2016)		X				X	X	(X)
Kolovou and van den Heuvel-Panhuizen(2010)			X			X	X	(X)
van den Heuvel-Panhuizen,Kolovou and Robitzsch(2013)			X			X	X	(X)
Bakker, van den Heuvel-Panhuizen and Robitzsch(2015)			X			X	X	
Ke(2014)				X		(X)		
Ke (2019)		X				X	(X)	
Lee and Chen (2009)	X					X	(X)	
Wijers, Jonker and Drijvers(2010)		X				X		
Wouters et al. (2015)					X			
Wouters et al. (2017)					X			

Studies with a similar shade of grey are part of the same research project. A parenthesised X indicates that the identified R&P phase was present to a lesser degree, implicit in the study or not part of the main affordance for mathematical reasoning. Note that two studies did not present evidence that we interpreted as being part of the R&P phases. This issue is addressed under the theme solving tasks

their strategies were winner strategies, supported by the teacher's scaffolding of their mutual and whole-class talks. While the students in Gresalfi (2015) and Gresalfi and Barnes (2016) engaged predominantly in consequential and not in mathematical justification, in Bakker et al. (2015), Kolovou and van den Heuvel-Panhuizen (2010) and van den Heuvel-Panhuizen et al. (2013), they justified their experimental data primarily by performing calculations.

Our analysis suggests that the main affordance *developing (winner) strategies* seems highly promising for students to learn to reason mathematically when the reasoning process is thoroughly framed didactically. One reason for this is that the object of student reasoning is closely tied to the rules of the game, and students must explore and identify underlying mathematical patterns to develop strategies to win the games. This affordance for exploration gives teachers opportunities to engage students in all three R&P phases, and this engagement is supported both by letting students play the game and by prompting their reflections on how the game is played.

Notably, the games *Lines* (Houssart & Sams, 2008) and *Frog Leap* (Lee & Chen, 2009) are not advanced in terms of graphics, narrative or other features normally associated with commercial digital games. Instead, their design is focused on making the mathematical structures of the games visible and encouraging experimentation with them. This is done through in-game features such as the possibility to rewind and fast-forward the game (Houssart & Sams, 2008) and the ability to edit relevant game settings, such as the number of frogs in *Frog Leap*. Such design features afford student experimentation and are a starting point for dialogue about the influence of different settings on students' gameplay.

In summary, the most promising DGBLEs for mathematical reasoning combine a game that supports students' exploration of mathematical relations, or highly specific game design features that target elements of the R&P cycle, and a dialogical framework that supports students' conjecturing and justification. This could take the form of students' gameplay in school or at home followed by classroom discussions (van den Heuvel-Panhuizen et al., 2013) or exploratory talk (Mercer, 1995) to structure dialogue around gameplay (Houssart & Sams, 2008), or from asking students to compare and discuss different results from a game with other students who have reached different results (Gresalfi & Barnes, 2016).

The positive findings from these studies highlight that a DGBLE aimed at using digital games to promote mathematical reasoning must consider the use of the game in relation to the classroom environment. In particular, classroom interactions among students, as well as between students and teachers, are vital for affording mathematical reasoning. In this regard, the integration of digital games in education mimics results from general studies on ICT that show that technology and pedagogy must be well integrated to be useful (e.g. Genlott & Grönlund, 2016), as well as outcomes suggesting that the teacher's role can both support students' gameplay and, at the same time, be in conflict with the game-based approach to learning (Vangsnes & Økland, 2015).

Game Design Affordances

In the analysis, we have identified how the DGBLEs afford mathematical reasoning. Three design features of the digital games afforded this in particularly positive ways: situation-based feedback, influencing game settings and NPCs.

First, situation-based feedback can lead to increased student verification in the justification phase (Kolovou & van den Heuvel-Panhuizen, 2010). The timing of feedback is essential: feedback provided after students deliver their solution does not lead them to revise their reasoning nor to consider alternative solutions. On the other hand, feedback provided early in the students' problem-solving process can create awareness of multiple solutions and increase student justification of their own solutions (Gresalfi & Barnes, 2016). Second, allowing students to experiment with changing game settings (Lee & Chen, 2009) can support them in engaging both in the exploration and in conjecturing phases (Kolovou & van den Heuvel-Panhuizen, 2010).

This second design feature makes the underlying structures of the game available for interaction, thereby affording student exploration and dialogue about patterns in the game, such as showing a second mathematical representation of the gameplay, displaying the co-ordinates for each move along with sequences of previous moves and making it possible to rewind gameplay (Houssart & Sams, 2008). Third, NPCs are a new feature of digital games that were implemented differently across the studies. The TA interacted with students' gameplay, learning from the students' input, which led students to judge the TA's conjectures and justifications (Pareto, 2014; Pareto et al., 2012). Another NPC was the expert computer-player that acted both as an inspiration for playing smartly and as an opponent that encouraged students to outsmart their opponent (Houssart & Sams, 2008). In addition, the second mayor in *Ander City* was an NPC, introduced as a part of the game narrative, to support students to see that problems could have more than one legitimate solution, which increased their justifications of their own solutions (Gresalfi, 2015). These three design features highlight the potential for the digital component of games to create other forms of interactions (compared with analogue games) that afford mathematical reasoning.

Approaches to Mathematical Reasoning

The reviewed studies defined mathematical reasoning in a variety of ways, which complicates comparison between them. At one end, Wouters et al. (2015, 2017) investigated *proportional reasoning skills*, without explicitly defining them, and with tasks that primarily seemed to support students in practicing skills and memorising facts. The two studies do not refer to exploration, conjecturing or justification, and their approach has little in common with the R&P cycle. At the other end, Ke (2014) defined mathematical reasoning as modelling, focusing on the relationships between developing, testing and debugging a program and the mathematics involved. She also used *abstract reasoning*, *content-specific reasoning*, *case-based reasoning*, *fundamental reasoning skill*, *everyday reasoning*, *quantitative reasoning*, *inductive and deductive reasoning*, *analytical reasoning*, *math reasoning* or simply *reasoning*, which makes it difficult to navigate the different concepts in the study.

Between these two ends, other studies have used, for instance, *procedural*, *conceptual*, *consequential* and *critical engagement* in mathematics to analyse students' activities (Gresalfi, 2015; Gresalfi & Barnes, 2016). While the last two of these considered mathematical reasoning to some extent, as consequential engagement involves exploring and predicting the impact of one's solutions in a game context and linking different solutions with different consequences, critical engagement captures the decision-making involved in problem solving, such as actively choosing tools and mathematically justifying one's choices.

Some authors distinguished between mathematical and other forms of reasoning: for example, Ke (2019) referred to crude reasoning and Pareto (2014) to casual reasoning. To add to the complexity, reasoning was also used to refer to logical thinking with implicit understandings of what this meant in the specific context: for example, *reasoning about area* (Wijers et al., 2010) or *deep-level reasoning* (Pareto, 2014). This complexity is underscored by what Yackel and Hanna (2003) describe as the sometimes-present implicit assumption that there is a universal agreement on what reasoning means, which Jeannotte and Kieran (2017) later clarified was not the case. These diversities in the studies' approaches to reasoning make comparison between them difficult.

One explanation for the diversity may be the skewed distribution of research fields in the studies. In our initial coding, we found that only four of the twenty-three authors in the studies were affiliated with mathematics education, while eight were from departments of general education, teaching and learning, and a further eight from departments of educational technology, computer education, computing sciences and instructional technology. Of the fourteen studies, eight were both published in journals outside of mathematics education and by authors outside this field. Only two studies were published in a mathematics education journal and had at least one author working in a mathematics education department. The former combination of both journals and authors from outside the field of mathematics education could prove problematic for a research field dedicated to understanding students' learning of mathematics. Such studies risk presenting results that are difficult to reconcile with those from the mathematics education community, especially if they do not take research from this field into account. We do not intend to suggest that this highly interdisciplinary field should only be restricted to mathematics education researchers. However, we do highlight the apparent lack of mathematics education researchers interested in this field, and that quality and relevance could be improved by promotion of the mathematics educational perspective in future studies. Ultimately, it would appear that special care needs to be given to understanding the domain of mathematics education, especially when broaching concepts such as mathematical reasoning, where even the current state of research within mathematics education is based on approaches so varied that comparison between them is difficult (Jeannotte & Kieran, 2017; Stylianides, 2016). In this review, we sought to mitigate this by being explicit about our own approach to mathematical reasoning.

In our reading of the selected studies, we found that three aspects helped us understand how mathematical reasoning was conceptualised. We therefore encourage future research on the relationships between digital games and students' mathematical reasoning to make these three aspects more explicit. The first is to define mathematical reasoning in terms that enable comparisons with existing research from mathematical education. The second is to distinguish between mathematical reasoning and reasoning in relation to games or contexts, such as consequential justification (Gresalfi & Barnes, 2016), as reasoning in and around the games does not need to be mathematical in nature. The third is to differentiate between mathematical reasoning and other forms of mathematical thinking, such as being procedurally engaged (Gresalfi, 2015) or solving tasks, in order to clarify that students can engage with the mathematics of a DGBLE without being engaged in mathematical reasoning.

Documenting Mathematical Reasoning

The documentation of students' mathematical reasoning and gameplay is a difficult task. The digital setting offers new possibilities for documentation, such as logging students' gameplay (Kolovou & van den Heuvel-Panhuizen, 2010), but without necessarily making documentation of mathematical reasoning easier. Across our selected studies, the task of documenting was approached differently. In total, ten studies (Bakker et al., 2015; Gresalfi & Barnes, 2016; Ke, 2019; Kolovou & van den Heuvel-Panhuizen, 2010; Lee & Chen, 2009; Pareto, 2014; Pareto et al., 2012; van den Heuvel-Panhuizen et al., 2013; Wouters et al., 2015, 2017) included some form of post-testing of mathematical learning. Additionally, some studies also identified mathematical reasoning by analysing transcriptions of students' dialogues and interactions with the DGBLEs (e.g. Gresalfi & Barnes, 2016; Houssart & Sams, 2008; Ke, 2014, 2019) or by coding students' written work in terms of mathematical and consequential justification (Gresalfi, 2015; Gresalfi & Barnes, 2016) or student strategies (Kolovou & van den Heuvel-Panhuizen, 2010). As a new means of documentation offered by the digital set-up, Pareto (2014) and Pareto et al. (2012) used the level of the TA as a proxy to indicate students' learning. However, being a proxy leads to uncertainty regarding what it means in relation to students' mathematical reasoning.

Regardless of the approach used, one issue in documenting student mathematical reasoning is that students can produce answers without using mathematical reasoning. For example, if students are asked to answer comparison and missing-value problems in the proportional reasoning domain, but not to explain how they arrived at their answer, it 'can lead to greatly overestimating a student's ability, since a correct answer can be generated from non-proportional reasoning' (Tourniaire & Pulos, 1985, p. 183). This suggests that post-tests determined to assess learning gains in terms of mathematical reasoning must enable students to explain their reasoning and not simply measure correct answers.

Moreover, the relationship between mathematical reasoning and the development of what Houssart and Sams (2008) call *a reasoned and winning approach* in gameplay was not always documented in the studies. There were clear intended affordances in the games that students would discover and explore properties and relations (Pareto et al., 2012), predict and reason about numbers (Pareto, 2014), discover relations between quantities (van den Heuvel-Panhuizen et al., 2013), investigate and interpret (Ke, 2019) and explore and consider statistical tools (Gresalfi, 2015). However, it was often difficult to judge whether and how these affordances were exploited by the students, as crucial elements of these processes were not explicitly documented in the respective studies.

Using the R&P cycle, we were able, in some cases, to conceive of the students' game choices as hypotheses or conjectures that they were forming in relation to current and future game stages. Nevertheless, the developments of these conjectures were often implicit. For instance, it is unclear how students received feedback on their answers to the TA or whether this feedback appeared only indirectly as consequences of their card choices in the game in Pareto (2014) and Pareto et al. (2012). This means that it is unclear how students can explain, validate or refute their hypotheses (answers) and be supported in developing strategies for playing well. As a result, a key issue for further research would be to make students' conjectures more explicit and to document more precisely when and how students engage in reasoning processes—for instance, by using the R&P cycle.

One challenge in making students' mathematical reasoning explicit in games is that, if students play against each other, cultural and social norms can encourage them to keep their strategies secret from other players rather than sharing them. One way to address this, which is afforded by digital games, is to position the students to compete with the computer-player or the environment rather than with each other.

Another consideration is that the DGBLEs affect the students' way of doing mathematics in unpredictable ways which can be difficult to document. Kolovou and van den Heuvel-Panhuizen (2010) suggest, for instance, that the game-generated feedback from the archery game led some students to verify answers in the post-test and create student-generated feedback. This leads to a question over whether the lack of mathematical reasoning outcomes is due to the DGBLEs not affording it or simply to insufficient documentation. As stated above, assessment of mathematical reasoning needs to include ways for students to explain their thinking to document their reasoning convincingly.

The challenge, therefore, is to design a DGBLE that affords explicit exploration, conjecturing and justification, while at the same time allowing for documentation of their occurrences.

Obstacles for Affording Mathematical Reasoning

Engaging students in mathematical reasoning is a challenging task (Nardi & Knuth, 2017; Stylianides, 2016; Stylianides & Stylianides, 2017). The use of digital games in mathematics education presents us with some specific obstacles. One idea is that digital games create engaging learning experiences (Kolovou & van den Heuvel-Panhuizen, 2010) and can therefore be used to teach students specific content. However, in order to afford this effectively, certain structures need to be present in the learning experience (Bakker et al., 2015). Other elements are best avoided when using games for learning mathematical reasoning, such as the continuous use of repetitive games, which does not seem to lead to new mathematical insight (Wouters et al., 2015, 2017). Indeed, this can create tensions in the gaming encounter (Goffman, 1961), because it affects the kinds of sense that can be made by the participants. If such structures are not sensitive to the nature of participation in the game, they risk ruining the fun that can be created in the gaming encounter.

In the reviewed studies, we interpret this issue to be seen in how the studies' designed affordances for playing and reasoning mathematically were sometimes different from how students played and reasoned about the games in reality. Such discrepancies were variously described in the studies. Bakker et al. (2015) reported, for instance, that more gameplay did not automatically lead to more learning, while Pareto (2014) emphasised that it was important that students try to *play well*, defined as, 'making good choices […] which in turn involves predicting and performing mental calculations as well as reasoning about numbers and computations' (p. 255). She also showed that the students could play a lot without trying to play well, and that they did not learn more simply by playing more.

On the other hand, van den Heuvel-Panhuizen et al. (2013) identified three student strategies when playing: *free playing*, *looking for answers* (i.e. to the associated tasks) and *exploring relationships*. Students who explored relationships scored significantly higher in a post-test than those who played freely, while students who looked for answers scored marginally higher than free-playing students. Hence, it is not only playing the game that leads to learning, but also particular ways of playing. Ke (2019) found that some students only explored and applied the targeted mathematical content when they valued mathematical reasoning as an efficient strategy. Otherwise, they circumvented mathematical content interaction and tried to outwit the environment through trial-and-error, estimation, guessing and crude reasoning: as such, they sometimes by-passed the mathematical task—actions that she described as content-irrelevant, gaming strategies and *careless gameplay* (p. 16).

These discrepancies can possibly be attributed to poor design of games or poor application of the games by teachers. We suggest that it also points to a lack of understanding of how students make sense of DGBLEs in the classroom. Terms such as *careless gameplay* (Ke, 2019) indicate a specific perspective that may not encompass how students' knowledge of how to play games from their leisure time influences what is afforded to them. An example is the opportunity to apply trial-and-error strategies that are sometimes highlighted as one potential for using GBL (and certainly as a valid form of interaction) in digital gameplay outside of school. However, we found that trial-and-error strategies needed to be applied to the learning content to be helpful in student learning (Ke, 2019; van den Heuvel-Panhuizen et al., 2013). This indicates that, even though trial-and-error strategies are part of how the students approach the games, they are not efficient for learning in the long run, especially when applied to content other than what was intended.

The results here show that games and DGBLEs afford many different actions that do not only involve learning content. The perceived affordance to play well from a mathematical educational perspective is not necessarily the same affordance to play well from the student's perspective. This calls for further investigation of what such games afford for the students and not only if they afford the expected mathematical interactions. In short, what does it mean to play a game well from the student perspective, and how does this relate to the designed games and teacher/researcher intentions?

Our analysis shows less engagement of students in the justification process in particular. Here, we suggest two possible reasons for this. The first is that the consequences of the player's actions in a digital game are often directly reflected in the game. This is in line with findings on the use of dynamic geometry systems, where it has been reported that compelling dynamic visualisations often provide sufficient empirical evidence to students, meaning they no longer feel the need for further justification (Sinclair & Robutti, 2013). Comparable to this, our results indicate that justification can become irrelevant when students play digital games, because the way in which the narrative or context changes or how the empirical exploration occurs can be experienced as sufficient justification, making it unnecessary, from the students' perspective, to engage further in argumentation to validate or refute conjectures.

In games with a winner, a student's logic may be 'If I won the game, there is no need to justify that what I did was correct, because my winning proves it to be correct'. It can be argued that the justification the students forward in such a situation is the actual enacting of game strategies during gameplay and, if the strategies work in the game, then there is no need for the students to justify them. Game moves that are self-explanatory in this fashion do not afford explicit justification, as the effect they have on the game state can be considered self-justifying.

The other reason relates to the social perspectives regarding justification (Stylianides et al., 2017), whereby both the need for and the process of justification involve considerable social meaning-making. This means that students (in part) justify based on how they interpret the social situation (Yackel & Cobb, 1996). In our review, this social aspect of justification was underlined by the fact that Houssart and Sams (2008)—the study that most explicitly engaged students in the entire R&P cycle—was also the study with the most explicit focus on students' explorative dialogue and social interactions. Furthermore, the game's design features that target justification do so by imitating forms of social interaction, such as the TA (Pareto, 2014; Pareto et al., 2012),which mimics a master–apprentice relationship, or the competing mayors in the Gresalfi (2015) study. In this respect, future DGBLE designers must consider that justification will most likely happen if students can also participate in justifications as part of social interactions or interactions that imitate these social aspects.

Conclusion

This review of mathematical reasoning and digital games in the interdisciplinary field of mathematics education and game-based learning shows a large diversity among the fourteen included studies in terms of the DGBLE affordances for mathematical reasoning offered to students. From our analysis, we draw six conclusions in relation to our research question.

First, the thematic analysis answers our research question by identifying five main affordances for mathematical reasoning: (1) *developing (winner) strategies* and figuring out how to play better than one's opponents; (2) *exploring an immersive* environment by solving problems and understanding the consequences in the world, scenario or narrative; (3) *experimenting* with different values of game settings and exploring their relationship; (4) *designing learning games* and creating content-specific learning games; (5) *solving tasks* in an appealing game context.

This shows that there are different ways to use digital games to support mathematical reasoning. However, *developing (winner) strategies* proved to be especially fruitful in affording all three phases of the R&P cycle, perhaps because the studies in regard to this theme most successfully combined designed game features with a dialogical pedagogical frame. Themes 2, 3 and 4 each carry specific potentials that could be developed further. We identified the fifth theme—*solving tasks*—as the least fruitful, in part because the studies here were aimed more at training in procedures than engagement in mathematical activity.

Second, we found no evidence in the studies that students learn to reason mathematically as a natural by-product of simply playing digital games, for instance, at home. The complex processes of mathematical reasoning are different from what is normally required to play a digital game well, which suggests that it requires a specific game design to afford mathematical reasoning, as well as a specific research design to document it. The analyses suggest that digital games are better suited to afford experimentation and (to some extent) conjecturing, while justification was afforded more through dialogues about the game and gameplay. Nonetheless, we identified highly promising design features that should be developed and investigated further, such as NPCs like the TA character, situation-based feedback and features enabling interaction with the underlying patterns of the games. The analysis shows that mathematical reasoning can be achieved through the scaffolding of both students' interactions with certain aspects of specific games and their reflections on these interactions.

Third, students' digital gameplay can support learning to reason mathematically if accompanied by tasks that can elicit their discovery of embedded mathematical relations in the game or, more effectively, through follow-up discussions in class on students' discoveries of mathematical relationships between different game settings. Generally, reasoning is not especially present when students play the games in these studies, but it is present in their reflections on the gameplay. Moreover, interaction with specialised designed features, such as the possibility to rewind a game or to experiment with game settings, affords reasoning. One issue, however, is that students' conjectures in gameplay are mostly hidden, and it must be afforded for the students to make them explicit. Design features necessary to do so include, for example, the TA or the competing mayors.

Fourth, our review shows that digital games and mathematical reasoning represent a small research niche, primarily conducted in the fields of educational technology and general education. This indicates that the field has yet to capture the interest of the broader mathematics education research field. To advance the field, more research is needed into whether and how DGBLEs can be designed to afford mathematical reasoning more explicitly, and how their affordances can be didactically framed to enhance students' engagement in the conjecturing and justification phases of the R&P cycle, in particular. One key issue here is that, without solid documentation of the students' reasoning process, it is difficult to judge whether and under which conditions mathematical reasoning occurs. Researchers in the field should take special care to define their use of mathematical reasoning and aim to document explicitly when and how such reasoning is evident.

Fifth, in the reviewed studies, the envisioned ways in which the students engaged in mathematical reasoning when playing well were seen primarily from a mathematical perspective, without considering how students would perceive playing well when using their everyday experience with games as a reference point. Even though exploration was afforded by many of the DGBLEs, this did not necessarily involve exploring the mathematical features of the game: certain students would merely experiment with the game as a whole and engage in different interactions with it, some of which did not lead to learning. To us, this indicates that students can perceive playing well differently from what the designers and researchers originally intended. Investigating what is afforded to the students when playing games in mathematics education seems crucial to understanding this further. Failing to do so risks excluding students for one of the primary reasons for introducing games in education in the first place, namely that they are fun for students.

Sixth, our review shows promising design features of digital games in DGBLEs in affording students' learning of mathematical reasoning, such as offering a place for exploration or a virtual world in which to immerse themselves and imitate social interactions. However, when using digital games to enhance mathematical reasoning, one must acknowledge that mathematical reasoning also has a social component, and that the teacher's role consequently remains pivotal. How DGBLEs afford students' learning of mathematical reasoning is closely linked both to the design of the games and to how they are played and reflected on in class. Designers of a game meant for the learning of mathematical reasoning should therefore consider which part of the R&P cycle is afforded by interactions with the game, with the teacher and with other students. At this point, the most promising way to afford justification appears to be through dialogue about the games in student groups, with teachers or in classroom discussions.

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Data Availability The reviewed studies are from the databases Education Research Complete, British Education Index, ERIC and Matheduc. The data generated during the study are available from the authors upon request.

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Declarations

Conflict of Interest The authors declare no competing interests.

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